How to Identify Types of Spatial Processes – An Example

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1. Introduction

Models in general are representations of real-world systems, that are used when the analysis of the system itself is not feasible (Frank 2006, p.29). "A mathematical model is an equation, or set of equations, whose solution describes the physical behaviour of a related physical system" (Logan 2004, p.1). Partial differential equation (PDE) models are one representative of mathematical models. They are used for simulations of continuous physical phenomena related to acoustics, fluid dynamics, electromagnetism, thermodynamics, etc. (Press, Flannery et al. 1986).

There is a large variety of spatial processes that are subject of different disciplines; examples are the flow of groundwater, the expansion of forest fires, the dispersal of seeds, the growth of cities, and the migration of grasshoppers. Given the large number of different spatial processes, the question posed in this research is if there are types of processes that occur repeatedly in space related problems. We approach this question by applying the approved knowledge on representing physical processes with PDEs to the spatial domain. The objective is to classify spatial processes and reveal similarities among spatial processes form different disciplines rather than to find models for spatial processes. The basic principles of this research were presented in (Hofer 2007).

In this contribution we discuss a concrete example of a partial differential equation and identify the characteristics of processes that can be modelled with this equation. Thereby we define one type of process that we can use later on for classifying spatial processes.

2. The constant velocity advection equation

Partial differential equations (PDE) model the change of a variable of interest that depends on more than one independent variable like space and time (Logan 2004). A PDE "can be read as a statement about how a process evolves without specifying the formula defining the process" (Encyclopædia Britannica 2007). Therefore, distinct problems may be described by the same PDE. Beckmann (1970), for example, showed that innovation diffusion, expenditure diffusion, and migration can be modeled with the same PDE, namely the diffusion equation.

The advection equation describes the bulk movement of particles in a transporting medium like water or air. This PDE is derived from a fundamental conservation law. The conservation of some quantity like mass, energy, or momentum, is an underlying principle of many PDE models of physical phenomena. The conservation law states, that the amount of change of a quantity in a certain region has to be balanced with the amounts coming in, going out, and being created or destroyed in that region (Logan 2004). There are different variants of the general advection equation. The equation discussed here is the 1-dimensional constant velocity advection equation without sources (Equation 1). This equation consists of two components (Logan 2004):

- the state variable *u(x,t)*, which defines the density of some quantity at location *x* at time *t* and
- the flux $\phi(x,t)$, which represents "the amount of the quantity that is crossing the section at *x* at time *t*" (Logan 2004, p.10). The flux ϕ in the constant velocity advection equation is proportional to the density *u*; the constant *c* is expressing constant speed (Equation 2).

$$u_t + cu_x = 0 \qquad (1)$$

$$\phi = c\,u \qquad (2)$$

$$u(x,0) = F(x) \qquad (3)$$

The initial condition for the advection equation, which defines the initial density of the quantity, is given by Equation 3 (Weatherley 2006). The constant velocity advection equation is a hyperbolic PDE. Hyperbolic PDEs model wave-like processes. In the case of the equation described here, the quantity that is modelled moves at constant speed *c* and does not deform. This kind of movement is known as travelling wave (Logan 2004). Fig. 1 shows the spatial density profile of a travelling wave.

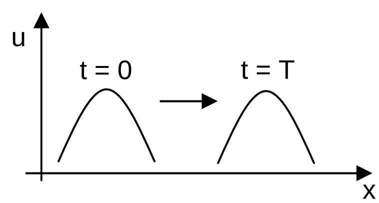

Figure 1. Spatial density profile of a travelling wave (after (Logan 2004)).

3. Identifying a certain type of process

Two examples for phenomena that can be described by the specific advection equation presented earlier are: the movement of a boat being carried downstream in a river and the motion of a dust particle drifting in the wind (Weatherley 2006). In both cases the transporting medium is assumed to show constant velocity. The function *F(x)* in the initial condition (Equation 3) can be understood as defining the shape of the boat or the dust particle (Weatherley 2006). In this section we discuss, why these examples can be related to the constant velocity advection equation. Table 1 lists the elements of the constant velocity advection equation and the analogues of the two examples.

Phenomena that can be described by the constant velocity advection equation connect a time- and space-dependent state variable and a constant velocity field. The focus on a constant velocity field is an obvious restriction of the applicability of this specific type of advection equation. Other processes that appear to have similar properties are, for example, pollutants being carried downstream in a river and a group of insects moving from one place to another. When pollutants are floating in a river, there is an advection and a diffusion process going on at the same time, because the pollutants also spread through the water while they are moving downstream. A group of insects has some internal behaviour and can therefore not be directly compared to a single object like a boat that is moving.

Constant velocity advection equation	Boat floating downstream	Dust particle in the wind
State variable *u* depending on space and time	Boat	Dust particle
Flux ϕ	Current of the river	Wind
Constant speed *c* of the flux	Current defined as a constant velocity field	Wind defined as having constant speed

Table 1: The elements of the PDE related to the analogues in the examples

4. Conclusions and further work

The long-term objective of this work is to identify types of processes that occur repeatedly in space related problems. The approach of this research is based on partial differential equations. In this contribution one specific PDE has been analysed and the characteristics of processes, which can be modelled with this equation, have been identified. Thereby, one example has been presented that shows how PDEs can be used to identify types of processes.

The example of the constant velocity advection equation indicated, that specific PDEs are applicable to special cases of phenomena. Further investigations of equations will shed light on the equations that have to be considered in a more general classification of spatial processes.

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