Modeling Visit Probability within Space-Time Prisms in Continuous Space and Time using Truncated Brownian Bridges

Ying Song, Harvey J. Miller

Department of Geography, University of Utah
260 S. Central Campus Dr., Rm. 270, SLC, UT 84112
Email: {ying.song; harvey.miller}@geog.utah.edu

1. Introduction

The space-time prism has been widely used to model individual movements with respect to both space and time. It demarcates all locations that a mobile object can occupy given the origin and destination anchors, the earliest origin departure time, the latest destination arrival time, and the maximum allowed travel velocity (Hägerstrand 1970, Lenntorp 1976). Although the prism boundary has been widely recognized and applied as a space-time accessibility measure (e.g. Miller 1991, Kwan 1998, Geurs and Wee 2004), little attention has been paid to the prism interior; all locations within the prism are considered to have undifferentiated properties such as the probability of being visited and the possible velocities at that location

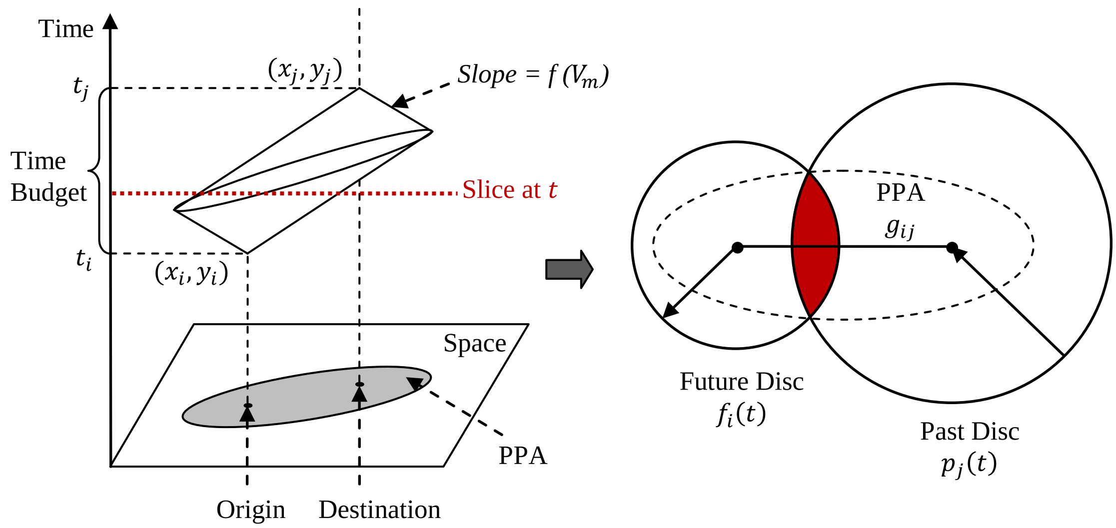
In a pioneering paper, Winter and Yin (2010a, 2010b) demonstrated that the probability to visit locations within the prism is not uniformly distributed; locations towards the central axis connecting two prism anchors are more likely to be visited than locations along the prism boundary. Although their results are intuitive, their theory, has some shortcomings. First, they simply applied the analog of result for discrete time and space to continuous time and space. Then, they made an assumption of the expected locations within the prim without any supportive fundamental theory. Finally, they applied an ad-hoc clipping method to confine the visit probability distribution within prism boundary.

In this paper, we develop methods to model visit probabilities within planar space-time prism based on Brownian Bridge (BB) theory for continuous stochastic process between two known values. First, we demonstrate the fundamental theory of space-time prism and BB and provide the mathematical foundation for modeling visit probability using BB theory. Then, we apply the truncated normal distribution to avoid the ad-hoc clipping and allow the prism boundary to arise naturally. We also modify the standard deviation and deal with the shifted mean after truncating. Finally, preliminary results are provided and discussed.

2. Background

2.1 Space-Time Prism in Classic Time Geography

The space-time prism (Hägerstrand 1970) captures spatial and temporal constraints on potential mobility and activity participation (Figure 1a). Analytically, the spatial extent of the prism with no stationary activity time at t (Figure 1b) is defined by intersection of two sets: a future disc $f_i(t)$ delimiting locations that can be reached from the first anchor and a past disc $p_j(t)$ delimiting locations that can reach the second anchor. The projection of the prism to planar space provides the potential path area (PPA) g_{ij}(Miller 2005).



(a) Geometric construction (b) Spatial extent at time t

Figure 1. Construction of the Space-time Prism.

2.2 Movements in Planar Space Modelled as Brownian Bridges

Brownian motion is a type of random movement based on continuous stochastic process (Andrei and Paavo 2002). A Brownian Bridge (BB) is Brownian motion anchored with known start and end values.

Individual movements in two dimensional planar space can be modeled as two-dimensional BBs. The two-dimensional BB method has been applied widely by ecologists to model animal movements between recorded locations (Horne et al. 2007, Benhamou 2011). Leaving origin (x_i, y_i) at t_i and arriving destination (x_j, y_j) at t_j, the visit probability at any time $t \in [t_i, t_j]$ is:

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} u_X(t) \\ u_Y(t) \end{bmatrix}, \begin{bmatrix} \sigma_X^2(t) & 0 \\ 0 & \sigma_Y^2(t) \end{bmatrix} \right) \quad (1)$$

$$u_X(t) = \frac{(t - t_i)x_i + (t_j - t)x_j}{t_j - t_i} \quad (2)$$

$$u_Y(t) = \frac{(t - t_i)y_i + (t_j - t)y_j}{t_j - t_i} \quad (3)$$

$$\sigma_{Std}^2(t) = \sigma_X^2(t) = \sigma_Y^2(t) = \frac{(t - t_i)(t_j - t)}{t_j - t_i} \quad (4)$$

Equations (2) and (3) support Winter and Yin's (2010b) assumption that the expected location moves with average speed along the axis connecting two prism anchors.

3. Methodology

3.1 Brownian Bridges Truncated by Dynamic Spatial Extent

For visit probabilities in continuous space-time, we avoid ad-hoc clipping of the Bivariate normal distribution and instead apply truncated distribution based on the prism spatial extent shown in Figure 1. A truncated distribution is a conditional distribution resulted from restricting the support of another distribution.

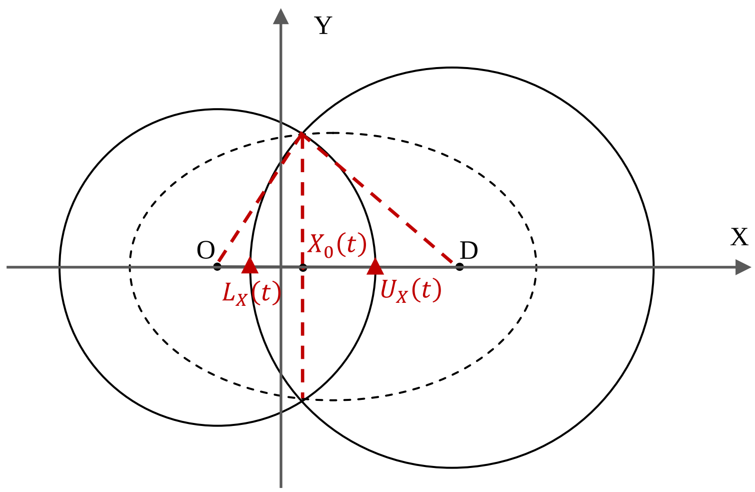


Figure 2. Truncating the Visit Probability Distribution at time $t \in [t_i, t_j]$.

For illustration purposes, we transform and rotate the coordination system so that the major axis of PPA is along x-axis and the mean location is located at (0,0). Therefore, we have the origin $(\bar{V}(t_i - t), 0)$ and the destination $(\bar{V}(t_j - t), 0)$, where $\bar{V}$ is the average speed between them calculated as:

$$\bar{V} = \sqrt{|x_j - x_i|^2 + |y_j - y_i|^2} \Big/ (t_j - t_i) \tag{5}$$

$X(t) \in [L_X(t), U_X(t)]$ follows the truncated normal distribution:

$$Tr(X(t)) \sim \frac{\mathcal{N}(0, \sigma_X^2(t))}{\Phi(U_X(t)) - \Phi(L_X(t))} \tag{6}$$

$$L_X(t) = (\bar{V} - V_m)(t_j - t) \tag{7}$$

$$U_X(t) = (V_m - \bar{V})(t - t_i) \tag{8}$$

where $\Phi(x)$ is the cumulative density function (CDF) of normal distribution $\mathcal{N}(0, \sigma_X^2(t))$ and $\mathcal{N}(0, \sigma_Y^2(t))$.Similarly, given $\tilde{X}(t) \in [L_X(t), U_X(t)]$, $Y(t) \in [L_Y(t), U_Y(t)]$ satisfies:

$$Tr\big(Y(t)|\tilde{X}(t)\big)\sim \frac{\mathcal{N}(0,\sigma_Y^2(t))}{\Phi\big(U_Y(t)\big)-\Phi\big(L_Y(t)\big)} \tag{9}$$

$$-L_Y(t)=U_Y(t)=\begin{cases}\sqrt{\left[V_m\big(t_j-t\big)\right]^2-\left[\bar{V}\big(t_j-t\big)-\tilde{X}(t)\right]^2}, \tilde{X}(t)<X_0(t)\\ \sqrt{[V_m(t-t_i)]^2-[\bar{V}(t-t_i)-\tilde{X}(t)]^2}, \tilde{X}(t)\geq X_0(t)\end{cases} \tag{10}$$

$$X_0(t)=\frac{\big(V_m{}^2+\bar{V}^2\big)\big(t_i+t_j-2t\big)}{2\bar{V}} \tag{11}$$

The probability to visit a location $(\tilde{X}(t),\tilde{Y}(t))$ within the spatial extent at time $t\in[t_i,t_j]$ can be calculated as:

$$Prob\big((\tilde{X}(t),\tilde{Y}(t))\big)=\frac{\varphi(\tilde{X}(t))}{\Phi\big(U_X(t)\big)-\Phi\big(L_X(t)\big)}\times\frac{\varphi(\tilde{Y}(t))}{\Phi\big(U_Y(t)\big)-\Phi\big(L_Y(t)\big)} \tag{12}$$

where $\varphi(\tilde{X}(t))$ and $\varphi(\tilde{Y}(t))$ are the probability density of $\mathcal{N}(0,\sigma_X^2(t))$ and $\mathcal{N}(0,\sigma_Y^2(t))$ at $\tilde{X}(t)$ and $\tilde{Y}(t)$ respectively.

3.2 Modified Variance and Shifted Mean

To capture the time-dependent variance and varying mobility levels, we modify the standard BB variance $\sigma_{Std}^2(t)$ in Equation (4), and define a modified variance based the time-varying prism bounds $U_X(t)$ and $L_X(t)$ and a mobility dispersion parameter δ:

$$\begin{aligned}\sigma_M(t)^2&=(V_m-\bar{V})^2\times(t-t_i)\times\big(t_j-t\big)\times\delta^2\\&=(V_m-\bar{V})^2\times\big(t_j-t_i\big)\times\sigma_{Std}^2(t)\times\delta^2\end{aligned} \tag{13}$$

In order to maintain consistency with BB theory, we also adjust the shifted mean caused by truncation. Given $\mathcal{N}(0,\sigma_X^2(t))$ and bounded range $[L_X(t),U_X(t)]$, the mean of truncated normal distribution is (Johnson et al., 1994):

$$\tilde{\mu}_X(t)=Mean\big(Tr(X(t))\big)=0+\frac{\varphi\big(L_X(t)\big)-\varphi\big(U_X(t)\big)}{\Phi\big(U_X(t)\big)-\Phi\big(L_X(t)\big)}\times\sigma_X(t) \tag{14}$$

Since $L_X(t)$ and $U_X(t)$ are not necessary to be symmetric to the mean, the truncated mean $\tilde{\mu}_X(t)$ may shift relative to the untruncated mean. Therefore, we modify the distribution in Equation (6) with $\mathcal{N}(\tilde{\mu}_X{}^0(t),\sigma_X^2(t))$, where $\tilde{\mu}_X{}^0(t)$ satisfies:

$$E(X(t))'=\frac{1}{\sqrt{2\pi}\times\sigma_X(t)}\times\int_{L_X(t)}^{U_X(t)}\left[e^{-\frac{(x-\mu_X)^2}{2\sigma_X(t)}}\times x\right]dx=0 \tag{15}$$

The bisection method can be applied to solve this equation.

4. Results and Discussion

Figure 3 shows distribution of visit probability at selected time $t_k \in [0,100]$ given the origin$(50,50)$, the destination $(250,250)$, and the maximum travel velocity $V_m = 5.0$. Yellow and red colors indicate higher visit probabilities. For any time t_k, the PPA arises naturally and, as expected, the probabilities are highest along the shortest path connecting two prism anchors.

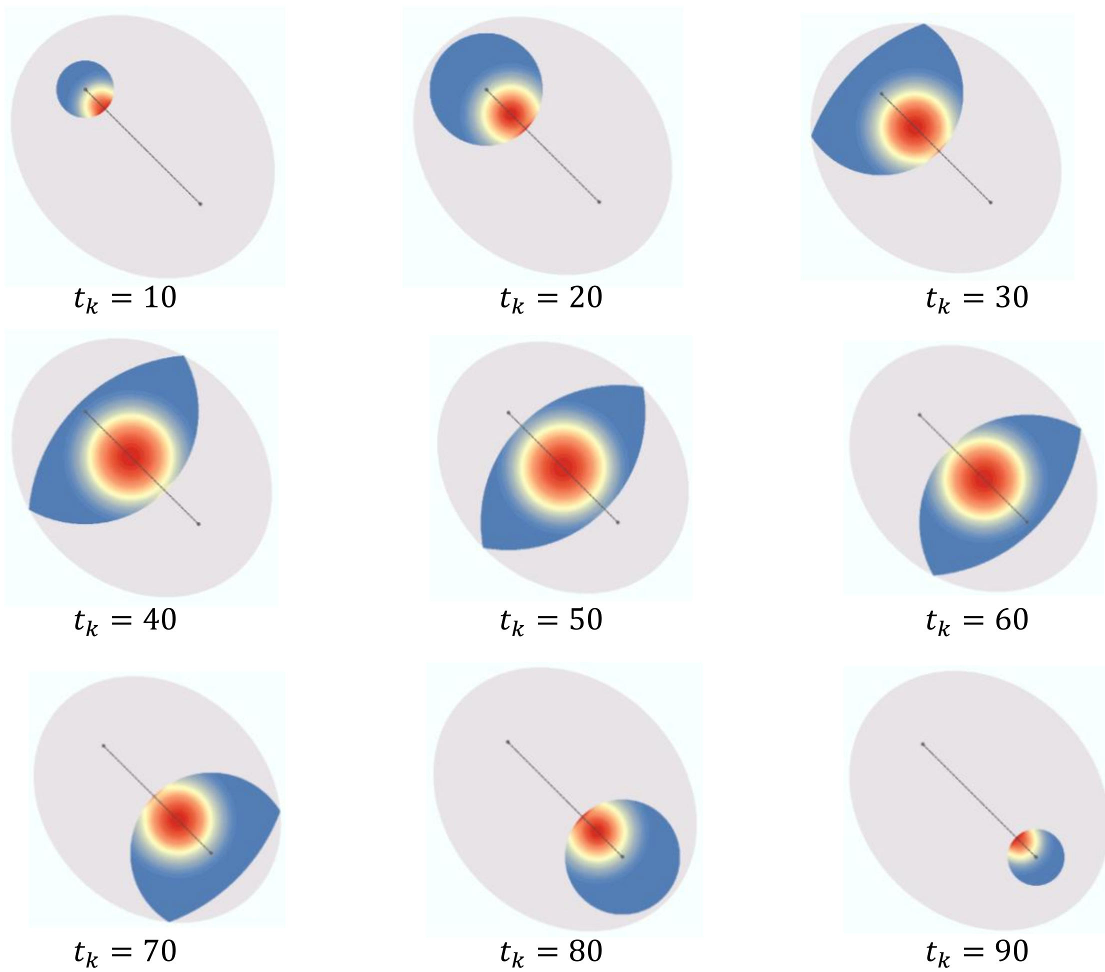


Figure 3. Distribution of Visit Probability at Selected Times

There are four parameters determine the visit probabilities simulated using BBs: i) the distance from the origin to the destination D_{ij}; ii) the time budget for travel T_{ij}; iii) the maximum allowed travel velocity V_m, and; iv) the dispersion parameter δ. Experimental results shown in Figure (4) - (7) show that the first three parameters determine the prism shape and the modified variance $\sigma_{Std}^2(t)$, while the last parameter is specific to $\sigma_{Std}^2(t)$.

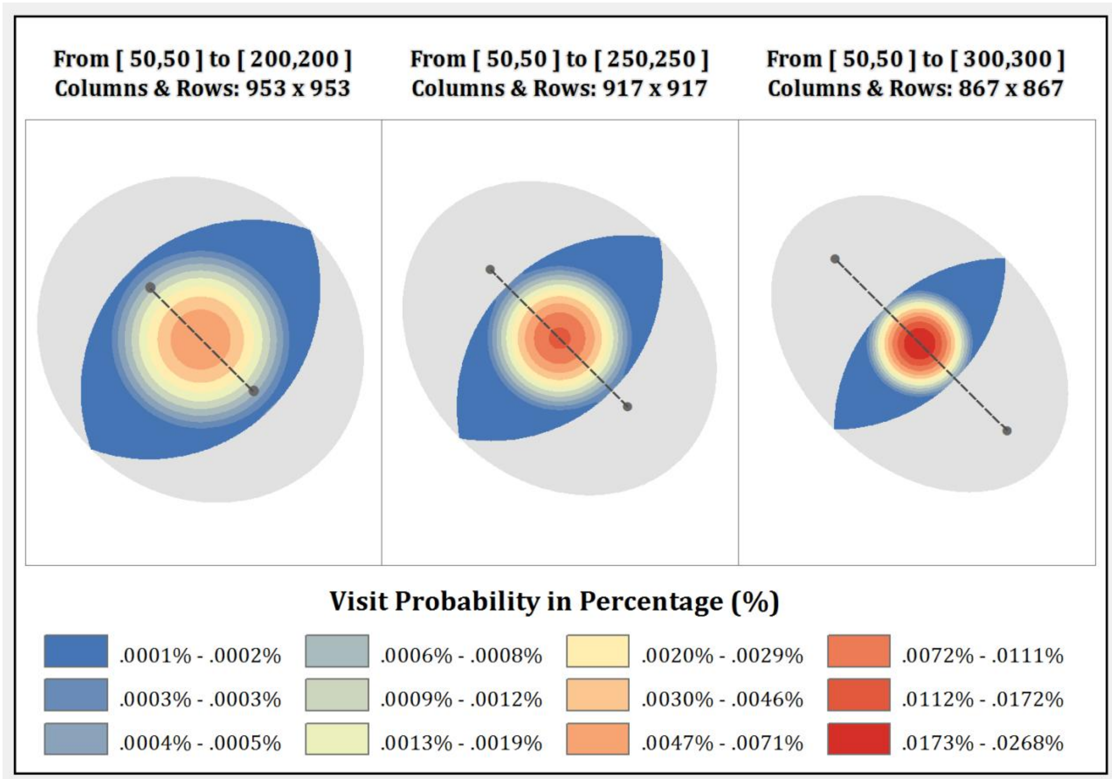


Figure 7: Impacts of Anchor Distance D_{ij} on Visit Probability Distribution

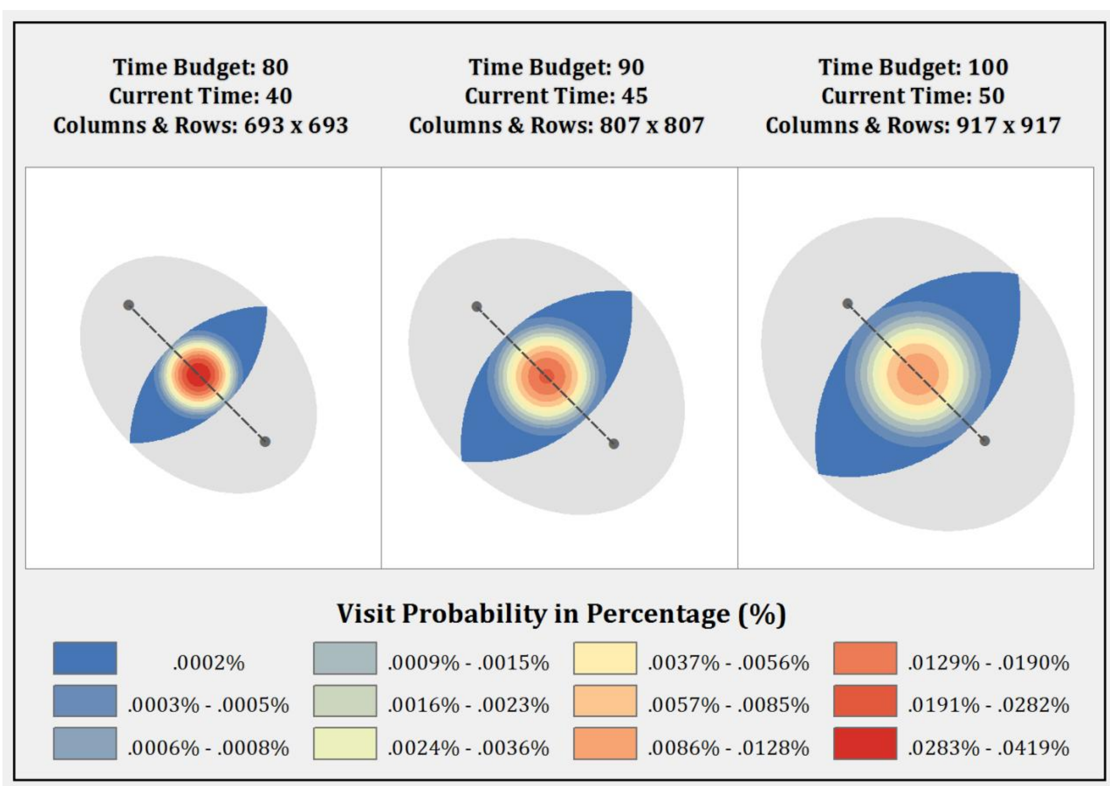


Figure 5 Impacts of Travel Time Budget T_{ij} on the Visit Probability Distribution

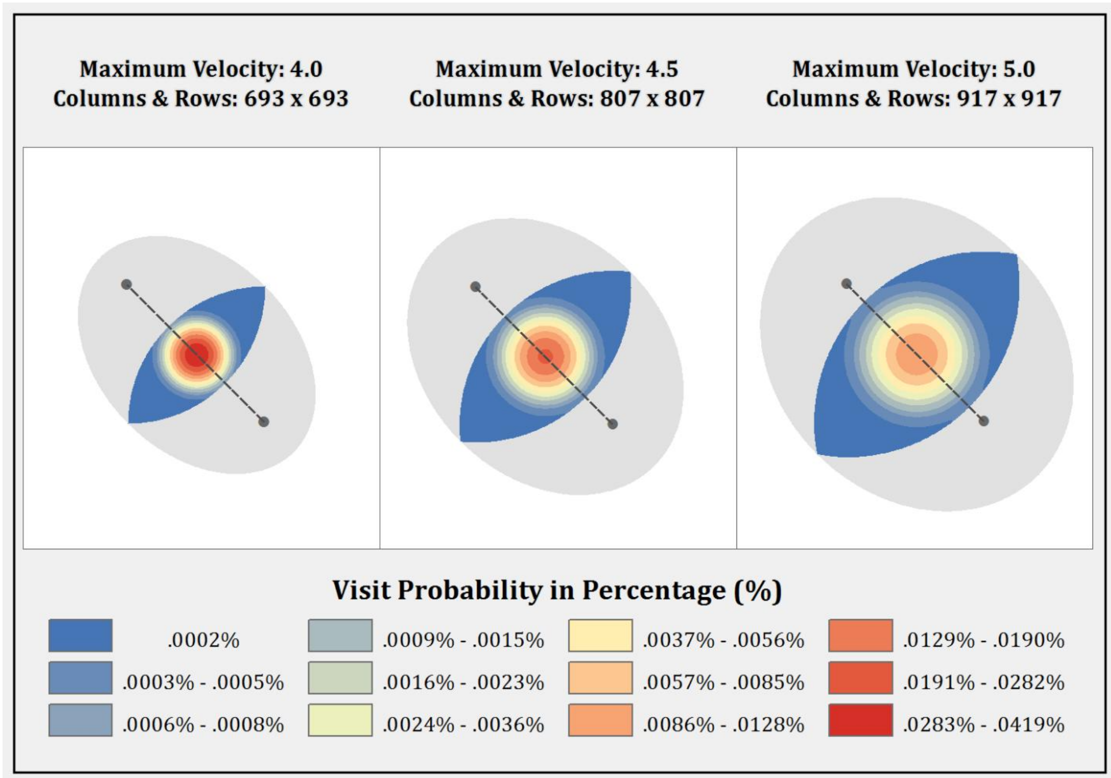


Figure 6 Impacts of Maximum Travel Velocity V_m on the Visit Probability Distribution

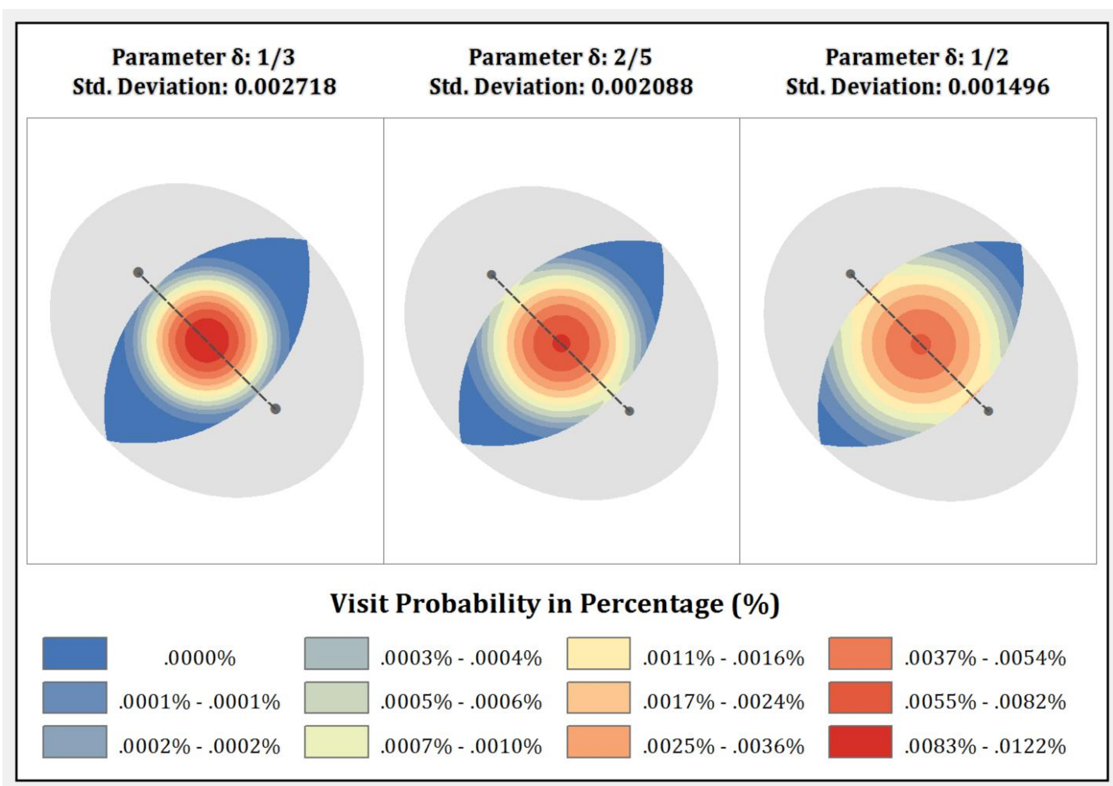


Figure 7 Impacts of Dispersion Parameter δ on Visit Probability Distribution

Compared to the Winter and Yin (2010a, b), our truncated BBs method is based on the fundamental principle and avoids ad hoc assumptions of expected locations as well as the artificial clipping of the Bivariate normal distribution. The simulation results show that our method allows the PPA to arise naturally. Varying the parameters describing movement possibilities generate visit probability distributions that match theoretical expectations.

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