A Novel Method for Earth Surface Modelling and Its Applications

T. X. Yue[1], N. Zhao[1], Z. P. Du[1], Z. M. Fan[1], C. L. Wang[1], C. F. Chen[2], D. J. Song[3], W. J. Shi[1],

[1]State Key Laboratory of Resources and Environment Information System, Institute of Geograpical Science and Natural Resources Research, 11A, Datun Road, Anwai, Beijing 100101, China.

Telephone: +86 10 64889041

Fax: +86 10 64889041

Email: Yue@lreis.ac.cn

Geomatics College, Shandong University of Science and Technology, Qingdao 266510, China

Telephone: +86 532 86057845

Fax: +86 532 86057848

Email: chencf@lreis.ac.cn

Institute of Policy and Management, Chinese Academy of Sciences, Beijing 100080, China

Telephone: +86 10 59358506

Fax: +86 10 59358608

Email: songdj@lreis.ac.cn

Ground observation can obtain high accuracy data at observation points, but observations at fixed positions are confined within some limited dispersal points and not able to directly calculate relative parameters at regional scale. Satellite remote sensing can frequently supply surface information of geographical processes and ecological processes, but remote sensing description is not able to directly obtain process parameters. Remote-sensing data can generate information about earth surface that is impossible from ground-based studies. However, maps derived from satellites observations are patchy and can not be used reliably as an independent source of information for earth surface monitoring because of the well know limitations of satellite retrievals, such as missing data for cloud-covered pixels (Emili et al. 2011). The most effective use of remote-sensing data is through its fusion with appropriate field investigation.

In terms of fundamental theorem of surfaces, a surface is uniquely defined by the first fundamental coefficients, about the details of the surface observed when we stay on the surface, and the second fundamental coefficients, the change of the surface observed from outside the surface. A high accuracy and speed method for surface modeling (HASM) has been developed initiatively to find solutions for error problem and slow-speed problem of earth surface modeling since 1986 (Yue 2011). HASM takes global approximate information (e.g. remote sensing images or simulation results) as its driving field and local accurate information (e.g. ground observation data or sampling data) as its optimum control constraints. Its output satisfies the iteration stopping criterion which is determined by application requirement for accuracy.

Earth's surface obeys the conditions of uniqueness, continuity, smoothness, and finiteness because the Earth's surface height, as a vertical coordinate, is restricted in space by the value of gravity and cannot be infinitely large or infinitely small. Attributes of the earth surface could be specially considered during the process of surface modeling include lines of relief discontinuity, steep and overhanging scarps, acute peaks, niches, caves, karst pits and sinkholes, as well as other elements violating the condition of smoothness. It is proven that the equation of Earth's surface can be formulated as (Kerimov 2009), $z = f(x, y)$ where z is an attribute value of the earth's surface at location (x, y).

For a surface $z = f(x, y)$, if it has continuous partial derivatives of first order, the first fundamental coefficients, E, F and G, can be formulated as

$$\begin{cases} E = 1 + f_x^{\ 2} \\ G = 1 + f_y^{\ 2} \\ F = f_x \cdot f_y \end{cases} \tag{1}$$

If $z = f(x, y)$ has continuous partial derivatives of second order, the second fundamental coefficients, L, M and N, can be formulated as,

$$\begin{cases} L = \dfrac{f_{xx}}{\sqrt{1 + f_x^{\ 2} + f_y^{\ 2}}} \\ N = \dfrac{f_{yy}}{\sqrt{1 + f_x^{\ 2} + f_y^{\ 2}}} \\ M = \dfrac{f_{xy}}{\sqrt{1 + f_x^{\ 2} + f_y^{\ 2}}} \end{cases} \tag{2}$$

The second fundamental coefficients reflect the local warping of the surface, namely its deviation from tangent plane at the point under consideration (Liseikin, 2004).

If the refined symmetric stencil is employed, HASM can be reformulated as

$$
\begin{cases}
\dfrac{-f_{i+2,j}^{(n+1)}+16f_{i+1,j}^{(n+1)}-30f_{i,j}^{(n+1)}+16f_{i-1,j}^{(n+1)}-f_{i-2,j}^{(n+1)}}{12h^2} \\
= (\Gamma_{11}^{1})_{i,j}^{(n)}\dfrac{f_{i+1,j}^{(n)}-f_{i-1,j}^{(n)}}{2h} + (\Gamma_{11}^{2})_{i,j}^{(n)}\dfrac{f_{i,j+1}^{(n)}-f_{i,j-1}^{(n)}}{2h} + \dfrac{L_{ij}^{(n)}}{\sqrt{E_{i,j}^{(n)}+G_{i,j}^{(n)}-1}} \\
\dfrac{-f_{i,j+2}^{(n+1)}+16f_{i,j+1}^{(n+1)}-30f_{i,j}^{(n+1)}+16f_{i,j-1}^{(n+1)}-f_{i,j-2}^{(n+1)}}{12h^2} \\
= (\Gamma_{22}^{1})_{i,j}^{(n)}\dfrac{f_{i+1,j}^{(n)}-f_{i-1,j}^{(n)}}{2h} + (\Gamma_{22}^{2})_{i,j}^{(n)}\dfrac{f_{i,j+1}^{(n)}-f_{i,j-1}^{(n)}}{2h} + \dfrac{N_{ij}^{(n)}}{\sqrt{E_{i,j}^{(n)}+G_{i,j}^{(n)}-1}} \\
\dfrac{f_{i+1,j+1}^{(n+1)}-f_{i+1,j}^{(n+1)}-f_{i,j+1}^{(n+1)}+2f_{i,j}^{(n+1)}+f_{i-1,j}^{(n+1)}-f_{i,j-1}^{(n+1)}+f_{i-1,j-1}^{(n+1)}}{2h^2} \\
= (\Gamma_{12}^{1})_{i,j}^{(n)}\dfrac{f_{i+1,j}^{(n)}-f_{i-1,j}^{(n)}}{2h} + (\Gamma_{12}^{2})_{i,j}^{(n)}\dfrac{f_{i,j+1}^{(n)}-f_{i,j-1}^{(n)}}{2h} + \dfrac{M_{ij}^{(n)}}{\sqrt{E_{i,j}^{(n)}+G_{i,j}^{(n)}-1}}
\end{cases}
\tag{3}
$$

where

$$
L_{i,j}^{(n)} = \frac{\dfrac{-f_{i+2,j}^{(n)}+16f_{i+1,j}^{(n)}-30f_{i,j}^{(n)}+16f_{i-1,j}^{(n)}-f_{i-2,j}^{(n)}}{12h^2}}{\sqrt{1+(\dfrac{f_{i+1,j}^{(n)}-f_{i-1,j}^{(n)}}{2h})^2+(\dfrac{f_{i,j+1}^{(n)}-f_{i,j-1}^{(n)}}{2h})^2}}
$$

$$
N_{i,j}^{(n)} = \frac{\dfrac{-f_{i,j+2}^{(n)}+16f_{i,j+1}^{(n)}-30f_{i,j}^{(n)}+16f_{i,j-1}^{(n)}-f_{i,j-2}^{(n)}}{12\mathrm{h}^2}}{\sqrt{1+(\dfrac{f_{i+1,j}^{(n)}-f_{i-1,j}^{(n)}}{2h})^2+(\dfrac{f_{i,j+1}^{(n)}-f_{i,j-1}^{(n)}}{2h})^2}}
$$

$$
M_{i,j}^{(n)} = \frac{\dfrac{f_{i+1,j+1}^{(n)}-f_{i+1,j}^{(n)}-f_{i,j+1}^{(n)}+2f_{i,j}^{(n)}+f_{i-1,j}^{(n)}-f_{i,j-1}^{(n)}+f_{i-1,j-1}^{(n)}}{2\mathrm{h}^2}}{\sqrt{1+(\dfrac{f_{i+1,j}^{(n)}-f_{i-1,j}^{(n)}}{2h})^2+(\dfrac{f_{i,j+1}^{(n)}-f_{i,j-1}^{(n)}}{2h})^2}}
$$

$$
E_{i,j}^{(n)} = 1+\left(\frac{f_{i+1,j}^{(n)}-f_{i-1,j}^{(n)}}{2h}\right)^2
$$

$$
G_{i,j}^{(n)} = 1+\left(\frac{f_{i,j+1}^{(n)}-f_{i,j+1}^{(n)}}{2h}\right)^2
$$

$$(\Gamma_{11}^{1})_{i,j}^{(n)} = \frac{E_{i+1,j}^{(n)} - E_{i-1,j}^{(n)}}{4hE_{i,j}^{(n)}}$$

$$(\Gamma_{11}^{2})_{i,j}^{(n)} = \frac{E_{i,j+1}^{(n)} - E_{i,j-1}^{(n)}}{4hG_{i,j}^{(n)}}$$

$$(\Gamma_{22}^{1})_{i,j}^{(n)} = -\frac{G_{i+1,j}^{(n)} - G_{i-1,j}^{(n)}}{4hG_{i,j}^{(n)}}$$

$$(\Gamma_{22}^{2})_{i,j}^{(n)} = \frac{G_{i,j+1}^{(n)} - G_{i,j-1}^{(n)}}{4hG_{i,j}^{(n)}}$$

$$(\Gamma_{12}^{1})_{i,j}^{(n)} = \frac{E_{i,j+1}^{(n)} - E_{i,j-1}^{(n)}}{4hE_{i,j}^{(n)}}$$

Let $f_{i,j} = \tilde{f}_{i,j}$ at the sampled point (x_i, y_j) in the computational domain, $(x_i, y_j) \in \Phi$, and $\Phi = \{(x_i, y_j, \tilde{f}_{i,j}) \mid 0 \le i \le \mathrm{I}+1, 0 \le j \le \mathrm{J}+1\}$ be the set of sampling points, then the matrix formulation of HASM can be expressed as,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \lambda\mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \lambda\mathbf{S} \end{bmatrix} \mathbf{z}^{(n+1)} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \lambda\mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{d}^{(n)} \\ \mathbf{q}^{(n)} \\ \mathbf{p}^{(n)} \\ \lambda\mathbf{k} \end{bmatrix} \tag{4}$$

Let $\overline{\mathbf{W}} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \lambda\mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \lambda\mathbf{S} \end{bmatrix}$ and $\overline{\mathbf{v}}^{(n)} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \lambda\mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{d}^{(n)} \\ \mathbf{q}^{(n)} \\ \mathbf{p}^{(n)} \\ \lambda\mathbf{k} \end{bmatrix}$, the equation set (4) is transformed as,

$$\overline{\mathbf{W}}\mathbf{z}^{(n+1)} = \overline{\mathbf{v}}^{(n)} \tag{5}$$

The coefficient matrix of HASM, $\overline{\mathbf{W}}$, is a symmetric positive-definite and large sparse linear matrix.

In terms of fundamental existing theorem for surfaces, if the first and second coefficients satisfy Gauss-Codazii equations, there exists a surface uniquely determined within a Euclidean displacement (Somasundaram 2005). The Gauss-Codazii equations can be transformed into,

$$\left(\varphi_{1y}-\phi_{2x}-\varphi_2 P-\phi_1 Q\right)^2+\left(\varphi_{2x}-\phi_{1y}-\varphi_1 Q-\phi_2 P\right)^2+\left(Q_x+P_y+\varphi_1\varphi_2-\phi_1\phi_2\right)^2=0 \tag{6}$$

where $\varphi_1=\frac{L}{\sqrt{E}}$; $\varphi_2=\frac{N}{\sqrt{G}}$; $P=\frac{\sqrt{E_y}}{\sqrt{G}}$; $Q=\frac{\sqrt{G_y}}{\sqrt{E}}$; $\phi_1=\frac{M}{\sqrt{G}}$; and $\phi_2=\frac{M}{\sqrt{E}}$.

Thus, we can design an iteration stopping criterion of the improved HASM as,

$$\left(\varphi_{1y}-\phi_{2x}-\varphi_2 P-\phi_1 Q\right)^2+\left(\varphi_{2x}-\phi_{1y}-\varphi_1 Q-\phi_2 P\right)^2+\left(Q_x+P_y+\varphi_1\varphi_2-\phi_1\phi_2\right)^2<EI \tag{7}$$

where EI is determined by the requirement of an application for simulation accuracy.

In this paper, principles and characteristics of HASM are described in details. We review robustness of HASM in its applications to constructing digital elevation model (Yue et al. 2007, Yue and Wang 2010, Yue et al. 2010a, b), filling voids of Data set of the Shuttle Radar Topography Mission (SRTM) (Yue et al. 2012), simulating climate change (Yue et al. 2011), and modeling surfaces of soil properties (Shi et al. 2009, 2011). In all these applications, HASM produced the highest accurate results comparing with the classical methods.

References

Emili E, Popp C, Wunderle S, Zebisch M, Petitta M, 2011, Mapping particulate matter in alpine regions with satellite and ground-based measurements: An exploratory study for data assimilation. *Atmospheric Environment*, 45: 4344-4353.

Kerimov IA, 2009, F-approximation of the Earth's surface topography. *Izvestiya, Physics of the Solid Earth*, 45(8): 719–729.

Liseikin VD, 2004, *A computational differential geometry approach to grid generation*. Springer, Berlin.

Somasundaram D, 2005, *Differential geometry*. Alpha Science International Ltd, Harrow, U.K.

Shi, W.J., Liu, J.Y., Song, Y.J., Du, Z.P., Chen, C.F., Yue, T.X. 2009. Surface modelling of soil pH. Geoderma 150(1-2):113-119.

Shi, W.J., Liu, J.Y., Du, Z.P., Stein, A., Yue, T.X. 2011. Surface modelling of soil properties based on land use information. Geoderma 162(3-4): 347-357.

Yue TX, Du ZP, Song DJ, Gong Y, 2007, A new method of surface modeling and its application to DEM construction. *Geomorphology*, 91(1-2): 161-172.

Yue TX, Chen CF, Li BL, 2010a, An adaptive method of high accuracy surface modeling and its application to simulating elevation surfaces. *Transactions in GIS*, 14(5): 615-630.

Yue TX, Song DJ, Du ZP, Wang W, 2010b, High-accuracy surface modelling and its application to DEM generation. *International Journal of Remote Sensing*, 31(8): 2205-2226.

Yue TX, Wang SH, 2010, Adjustment computation of HASM: a high-accuracy and high-speed method. *International Journal of Geographical Information Science*, 24(11): 1725 - 1743.

Yue TX, 2011, *Surface modelling: high accuracy and high speed methods*. CRC Press, New York.

Yue TX, Fan ZM, Chen CF, Sun XF, Li BL, 2011, Surface modelling of global terrestrial ecosystems under three climate change scenarios. *Ecological Modelling*, 222: 2342-2361.

Yue TX, Chen CF, Li BL, 2012, A high accuracy method for filling SRTM voids and its verification. *International Journal of Remote Sensing*, 33(9): 2815-2830.